Metaplectic formulation of linear mode conversion

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The metaplectic formulation of linear mode conversion is presented. We begin by discussing the connection between wave operators in weakly inhomogeneous media, their symbols, and related pseudodifferential operators. A brief summary of WKB theory and Hamiltonian ray dynamics is given. In regions where mode conversion occurs, the WKB approximation breaks down and must be replaced by an appropriate local approximation. This is done by expanding in a Taylor series about the degenerate region and keeping only the leading-order terms. At leading order a linear canonical transformation on the ray phase space can be performed which brings the system into a simpler form. This linear canonical transformation induces a unitary transformation, called a *metaplectic transformation,* in the wave function's Hilbert space. This is a generalization of the Fourier transformation. The advantage of metaplectic techniques over Fourier techniques lies in the wider range of transformations available to simplify the problem. We show how to construct the S matrix, relating incoming and outgoing waves, and the Wigner tensor. We examine the Wigner function in detail with particular attention to its asymptotic properties.

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I. INTRODUCTION AND MOTIVATION

A. Background remarks

In the present paper we discuss the application of metaplectic techniques to the analysis of linear mode conversion. The metaplectic approach is essentially a generalization of the Fourier transform. This approach to linear conversion first appeared in Ref. [1]. For an excellent review of metaplectic techniques and their application to semiclassical wave mechanics we refer the reader to Ref. [2]. Some of these topics are also covered in Refs. [3–5]. The present paper is organized as follows: In Sec. I B we discuss linear wave propagation in inhomogeneous media, introducing the basic integral form of the wave equation we will use. In Sec. I C we introduce the Weyl symbol of an operator and its associated pseudodifferential operator. We also briefly examine the transformation properties of the pseudodifferential operator under changes of representation.

In Sec. II we examine the phenomenon of linear conversion using the phase-space picture. The symbol is a function of the space-time position $(\mathbf{x},t)$ the wave number $\mathbf{k}$, and frequency ω. This leads naturally to the introduction, in Sec. II A, of the *wave phase space.* Away from the conversion region we assume that the WKB approximation is valid for each mode, leading to Hamilton's equations for their respective rays, as discussed in Sec. II B. In Sec. II C we examine the degenerate region and develop a local approximation to the pseudodifferential operator by expanding the Weyl symbol in a Taylor series. In Sec. II D we discuss linear canonical transformations, which will be of use in simplifying the conversion problem.

In Sec. III we discuss how linear canonical transformation of the ray phase space are associated with unitary transformations in the wave Hilbert space. These unitary transformations are the metaplectic transformations. In Sec. III A we point out the analogy between metaplectic operators, induced by linear canonical transformations, and the more familiar rotation operators induced by rotations in 3-space. In Sec. III B we construct the explicit matrix elements for the metaplectic operator induced by a given linear canonical transformation, giving an example of their use in changing representation.

There is no new material in these first three sections, However, many of these topics are not well covered in the physics literature; therefore we have included a brief discussion of them for completeness and to ensure the clarity of the present discussion. Full details are available in the references.

The main contribution of the present work is in Sec. IV. Here we show how metaplectic techniques can be used to advantage in the linear conversion problem. In Sec. IV A we discuss the choice of representation which simplifies the problem and its relation to the boundary conditions. Sec. IV B we present the explicit solutions to the wave equation in the conversion region and construct the S matrix. In Sec. IV C we evaluate the Wigner tensor and discuss its asymptotic properties. Some of these results were discovered independently by Littlejohn and Flynn [4]. However, we use a different representation for the wave field in the conversion region, and in addition we consider the behavior of the Wigner function in the conversion region in some detail. Balazs and Voros [6] have considered the nature of the Wigner function in tunneling regions, a related problem.

B. Waves in weakly inhomogeneous media

Consider linear wave propagation in a weakly inhomogeneous and slowly-time-varying medium. In many situations of physical interest the medium will support more than one type of wave mode. The general form of such wave equations is

$$\int dx' \mathbf{D}(x,x')\Psi(x') = \int dx' \sum_{\beta=1}^{N} D_{\alpha\beta}(x,x')\psi_\beta(x') = 0 \ , \tag{1.1}$$

where Ψ is an N-component wave function, $\mathbf{D}$ is an $N \times N$ matrix kernel, and x and x' are space-time coordinates [for example, $x = (\mathbf{x}, t)$]. We are particularly interested in finding WKB-type solutions to Eq. (1.1), i.e., solutions expressible as a rapidly varying phase function and a slowly varying amplitude.

C. The Weyl symbol and pseudodifferential operators

As a first step, we now introduce the *Weyl symbol* of the wave operator. The symbol is directly related to the dispersion relation, as we shall discuss in Sec. II. For a complete discussion we refer the reader to Ref. [5]. The Weyl symbol of $\mathbf{D}(x,x')$ is defined as

$$\mathbf{D}(y,k) \equiv \int d^4s \, e^{-ik\cdot s} \mathbf{D}\left[y + \frac{s}{2}, y - \frac{s}{2} \right]$$

$$y = \frac{x+x'}{2}, \quad s = x - x' \ . \tag{1.2}$$

Here, and in what follows, all integrals extend from $-\infty$ to $+\infty$ unless otherwise stated. k is the four-vector $(\mathbf{k}, -\omega)$ and $k \cdot s = \mathbf{k}\cdot\mathbf{s} - \omega s_0$. Notice that the symbol is a function of both the space-time position y and the four-vector k. We use the Weyl definition of the symbol, as opposed to other possible definitions [2,5,7], because the related pseudodifferential operator (to be defined in a moment) is symmetric in the position and momentum operators. Hence, when taking semiclassical limits, the position and momentum appear on an equal footing as desired in the Hamiltonian formulation.

We now introduce the pseudodifferential operator associated with the Weyl symbol. We will do this in a manner that is independent of representation, since this will be useful to us later. The definition is essentially that given by Weyl [8]. First take the Fourier transform in both x and k of the Weyl symbol:

$$\overline{\mathbf{D}}(\sigma,\tau) \equiv \frac{1}{(2\pi)^8} \int d^4y \, d^4k \, e^{-i(\sigma\cdot y + \tau\cdot k)} \mathbf{D}(y,k) \ .$$

The overbar signifies the Fourier transform. The pseudodifferential operator $\mathcal{D}$ associated with the symbol $\mathbf{D}(x,k)$ is

$$\mathcal{D}(\hat{x},\hat{k}) = \int d^4\sigma \, d^4\tau \, \overline{\mathbf{D}}(\sigma,\tau) \exp[i(\sigma\cdot\hat{x} + \tau\cdot\hat{k})] \ . \tag{1.3}$$

There are subtle convergence issues associated with these objects. A discussion of these issues is beyond the scope of this paper and we refer the interested reader to Ref. [9] and citations therein.

We have introduced the caret in (1.3) to denote the position and momentum operators. More precisely they are vectors of operators. For example, in the x representation $\hat{x}^\mu$ is multiplication by the four-vector x^μ while $\hat{k}_\nu$ has components

$$\hat{k}_\nu = -i\partial_\nu = -i(\nabla, \partial_t) \ .$$

Using this notation, we can write the full set of commutation relations in the compact form

$$[\hat{x}^\mu, \hat{k}_\nu] = i\delta^\mu_\nu \ , \tag{1.4}$$

where δ^μ_ν is the four-dimensional Kronecker delta function.

We can now rewrite Eq. (1.1) in the abstract form

$$\mathcal{D}(\hat{x},\hat{k})|\Psi\rangle = 0 \ . \tag{1.5}$$

The advantage of rewriting Eq. (1.1) in this more abstract form is that we can choose whatever representation is most convenient to analyze the problem. For example, to write Eq. (1.5) in the x representation, we first act from the left with $\langle x|$ and insert a complete set of states to find

$$\int d^4x' \langle x|\mathcal{D}(\hat{x},\hat{k})|x'\rangle\langle x'|\Psi\rangle \equiv \mathbf{D}(x,-i\partial)\Psi(x) = 0 \ .$$

Using Eq. (1.4) and

$$\langle x|\hat{x}|x'\rangle = \delta(x-x'), \quad \langle x|\hat{k}|x'\rangle = \delta(x-x')(-i\partial) \ ,$$

we have

$$\mathbf{D}(x,-i\partial) = \int d^4\sigma \, d^4\tau \, \overline{\mathbf{D}}(\sigma,\tau) \exp[i\sigma\cdot x + \tau\cdot\partial] \ .$$

Suppose we now change representation via an arbitrary unitary transformation

$$\hat{x}' \equiv U\hat{x}U^\dagger, \quad \hat{k}' \equiv U\hat{k}U^\dagger \ .$$

The pseudodifferential operator defined in Eq. (1.3) has the following simple transformation rule [9]:

$$U\mathcal{D}(\hat{x}\hat{k})U^\dagger = \mathcal{D}(\hat{x}',\hat{k}') \ , \tag{1.6}$$

as can be shown from the definition (1.3) using the unitarity of U. We shall find this transformation property useful in later sections when we examine the metaplectic transformations, a subgroup of the unitary transformations.

II. PHASE-SPACE PICTURE OF LINEAR MODE CONVERSION

A. The ray phase space

Suppose the Nth mode is nondegenerate in the sense that its dispersion relation is independent of the other $N-1$ modes. In such a case it is possible to carry out a reduction, i.e., to develop separate dynamical equations governing the single Nth mode and the other $N-1$ modes (see Ref. [5] for a complete discussion). In the generic case, all N modes are nondegenerate and, by induction, we can reduce Eq. (1.1) to N separate wave equations, one for each independent mode.

Degeneracy is nongeneric, hence it usually occurs only in special regions and involves only a single pair of modes. In such regions the full reduction procedure cannot be carried out and, generically, leads to a two-component wave equation for the two modes taking part in the conversion

$$\int d^4x' \mathbf{D}'(x,x')\mathbf{Z}(x')=\mathbf{0} . \tag{2.1}$$

Here $\mathbf{D}'(x,x')$ is a 2×2 matrix kernel and $\mathbf{Z}(x')$ is a two-component wave field. In the following sections we will be concerned only with this 2×2 problem; therefore we will drop the prime on the reduced kernel. Following the discussion in the preceding sections let us consider the symbol of the reduced kernel $\mathbf{D}(x,k)$:

$$\mathbf{D}(x,k)=\begin{bmatrix} D_a(x,k) & \eta'(x,k) \\ \eta'^*(x,k) & D_b(x,k) \end{bmatrix} . \tag{2.2}$$

We have written the system in the above manner because we wish to make the following assumption: the nature of the physics is such that the wave propagation can be separated into the two polarizations determined by $D_aZ_a=0$ (the a polarization) and $D_bZ_b=0$ (the b polarization) and a coupling η', which is small in some suitable sense. If the original wave dynamics is dissipationless, then the operator is Hermitian, implying that the symbol is also [10]. Hence D_a and D_b are real functions of x and k, called *dispersion functions,* while η' is in general complex (η'^* is the complex conjugate). The behavior of solutions to Eq. (2.1) depends strongly on whether the dispersion functions D_a and D_b can vanish simultaneously. In the generic case these functions are independent, implying that the conditions $D_a(x,k)=0$ and $D_b(x,k)=0$ will be satisfied only in a restricted region of the phase space, if it is satisfied at all. We assume there exists such a region. The requirements $D_a=0$ and $D_b=0$ involve two relations between eight variables, thus it specifies a six-dimensional submanifold of the phase space, called the *mode-conversion manifold,* which we denote by M_6. More precisely,

$$M_6\equiv\{(x,k)\colon D_a(x,k)=0, D_b(x,k)=0\} . \tag{2.3}$$

We now consider separately two distinct regions of phase space: the region far from M_6, the *nondegenerate* region, and the region near M_6, the *degenerate* region.

B. Nondegenerate regions of phase space

1. Local dispersion relations

The symbol is a function of both x and k. This suggests that the appropriate space in which to consider the wave dynamics is the eight-dimensional *wave phase space* (x,k). The wave function $\mathbf{Z}(x)$ is a function only of x (in the present x representation) and therefore is not a phase-space quantity. We can relate it to a phase-space object, however, via the introduction of the *Wigner tensor* [2,5,11]. The Wigner tensor of $\mathbf{Z}(x)$ is defined as

$$\mathsf{W}(x,k)\equiv\int d^4s\, e^{-ik\cdot s}\mathbf{Z}(x+\tfrac{1}{2}s)\mathbf{Z}^\dagger(x-\tfrac{1}{2}s) , \tag{2.4}$$

or in component form

$$\mathsf{W}_{jk}(x,k)\equiv\int d^4s\, e^{-ik\cdot s}Z_j(x+\tfrac{1}{2}s)Z_k^*(x-\tfrac{1}{2}s) .$$

Thus it is constructed in a manner analogous to that of the symbol. An important property of symbols is that they do *not* obey the same multiplication rules as their associated operators. Consider two operators $\hat{A}$ and $\hat{B}$ and their product $\hat{C}\equiv\hat{A}\hat{B}$. If $A(x,k)$, $B(x,k)$, and $C(x,k)$ are their associated symbols, then it is possible to show [11] that

$$C(x,k)=A(x,k)e^{(i/2)\mathcal{L}}B(x,k) , \tag{2.5}$$

where $\mathcal{L}$ is the Janus operator, defined as

$$\mathcal{L}\equiv\frac{\overleftarrow{\partial}}{\partial x}\cdot\frac{\overrightarrow{\partial}}{\partial k}-\frac{\overleftarrow{\partial}}{\partial k}\cdot\frac{\overrightarrow{\partial}}{\partial x} .$$

The arrows over the operators indicate in which direction they act. In particular, if F and G are any two functions on phase space we have

$$F\mathcal{L}G\equiv\{F,G\}_8=\frac{\partial F}{\partial x}\cdot\frac{\partial G}{\partial k}-\frac{\partial G}{\partial x}\cdot\frac{\partial F}{\partial k} ,$$

with $\{F,G\}_8$ the Poisson bracket on the eight-dimensional phase space:

$$\{F,G\}_8\equiv\frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial k_\mu}-\frac{\partial F}{\partial k_\mu}\frac{\partial G}{\partial x^\mu} . \tag{2.6}$$

We wish to use the symbol product rule (2.5) to analyze Eq. (2.1). Following Ref. [11] we multiply Eq. (2.1) from the right by $\mathbf{Z}^\dagger(x'')$

$$\int d^4x' \mathbf{D}(x,x')\mathbf{Z}(x')\mathbf{Z}^\dagger(x'')=\mathbf{0} .$$

Using Eq. (2.5) we find

$$\mathbf{D}(x,k)e^{(i/2)\mathcal{L}}\mathbf{W}(x,k)=\mathbf{0} . \tag{2.7}$$

In nondegenerate regions we can restrict our attention to WKB-type solutions, in which case we assume that $\mathbf{W}(x,k)$ is a slowly varying function of *both* x and k (it is an envelope function). This allows us to develop an asymptotic series in powers of the Janus operator:

$$e^{(i/2)\mathcal{L}}=1+\frac{i}{2}\mathcal{L}+\cdots .$$

At first order this gives

$$\mathbf{D}(x,k)\mathsf{W}(x,k)=\mathbf{0} . \tag{2.8}$$

If the wave is dissipationless, then $\mathbf{D}$ is Hermitian. If the wave is weakly dissipative, then we assume that the leading-ordering behavior is dissipationless [11]. Taking the Hermitian conjugate of (2.8) we get $\mathsf{W}\mathbf{D}=0$. Hence $\mathbf{D}$ and W commute and they can be simultaneously diagonalized. The diagonal terms are then of the form

$$D_n(x,k)\mathsf{W}_n(x,k)=0 \quad (n=a,b) . \tag{2.9}$$

Thus far from the mode-conversion region nontrivial solutions to Eq. (2.9) can be found only in regions of the phase space where either $D_a(x,k)=0$ or $D_b(x,k)=0$. These conditions define the *dispersion manifolds* in the phase space. Since $D_a(x,k)=0$ is one relation among eight variables its dispersion manifold is seven dimensional (respectively for $D_b=0$). The condition $D_n=0$ can be solved for the frequency ω, leading to

$$D_n(x,k)=0 \rightarrow \omega=\Omega_n(x,\mathbf{k}) ;$$

in other words the mode n must satisfy a *local dispersion relation.* In general there may be multiple branches to the dispersion relation.

2. *The ray equations*

Let us assume that the leading order condition in Eq. (2.7) is satisfied. The next order is $\mathbf{D}\mathcal{L}\mathsf{W}=\{\mathbf{D},\mathsf{W}\}_8$. Consider one of the diagonal terms

$$\{D_n,\mathsf{W}_n\}_8=0 \quad (n=a,b) . \tag{2.10}$$

Equation (2.10) has the form of Hamilton's equations. We can see this more clearly if we introduce a parameter σ_n, called the *ray orbit parameter* [5]. Equation (2.10) can be recast as

$$\frac{d\mathsf{W}_n}{d\sigma_n}=\frac{dx}{d\sigma_n}\cdot\frac{\partial \mathsf{W}_n}{\partial x}+\frac{dk}{d\sigma_n}\cdot\frac{\partial \mathsf{W}_n}{\partial k}=\{D_n,\mathsf{W}_n\}_8=0 .$$

The total derivative in σ_n represents the derivative following a ray on the surface $D_n=0$. The *ray equations* are

$$\frac{dx^\mu}{d\sigma_n}=-\frac{\partial D_n}{\partial x_\mu} \quad \frac{dk_\mu}{d\sigma_n}=\frac{\partial D_n}{\partial x^\mu} . \tag{2.11}$$

In particular Eqs. (2.11) can be used to relate the time t to the ray orbit parameter σ_n:

$$\frac{dt}{d\sigma_n}=\frac{\partial D_n}{\partial \omega}\equiv\partial_\omega D_n ,$$

which can, in general, be positive or negative. However, in order for σ_n to give a parametrization of the entire orbit we must have $dt/d\sigma_n\neq 0$ everywhere along the ray. In general $\partial_\omega D_n$ is independent of D_n, therefore the condition for them to be simultaneously zero is a restriction to a six-dimensional submanifold. In the general case it is possible that some rays may encounter such a region. However, this is not the case, for example, in the class of physically important wave equations with dispersion functions of the form $D_n(x,k)=\omega^m-\Omega_n^m(\mathbf{x},\mathbf{k})=0$ $(m=1,2,\ldots)$. In this case $dt/d\sigma_n=2m\omega^{m-1}$ and $d\omega/d\sigma_n=0$ [by Eq. (2.11)]. Thus we see that $dt/d\sigma_n\neq 0$, unless we choose $\omega=0$ as an initial condition. There may, however, be physically interesting cases where rays propagate into regions where $dt/d\sigma_n$ vanishes. In the present work we will assume that σ_n can be used as an orbit parameter for the entire ray.

This concludes our discussion of the nondegenerate "outer" region of the phase space. We now turn to the degenerate region.

C. Degenerate regions of phase space and linear mode conversion

Taylor expansion of the Weyl symbol about the degenerate manifold

In degenerate regions of the phase space many results of Sec. II B are no longer valid. In particular (1) the WKB approximation breaks down, (2) the wave fields are no longer restricted to dispersion manifolds, and (3) wave disturbances no longer propagate along well-defined ray trajectories. It is important to recognize, however, that the ray equations (2.11) are still well defined and will allow us to connect incoming and outgoing solutions. Our goal is to construct inner solutions to Eq. (2.1) which are valid in the region surrounding the conversion manifold and then match them asymptotically to the incoming and outgoing solutions away from the conversion region. These outer solutions are constructed using ray tracing methods as described in Sec. II B.

The dispersion functions D_a and D_b vanish simultaneously on the six-dimensional conversion manifold M_6. Consider a ray of mode a entering the conversion region (Fig. 1). It will puncture the dispersion manifold for mode b somewhere in M_6. Call this point (x_0,k_0). Taylor expand the Weyl symbol about the conversion point

$$\mathbf{D}(x,k)=\mathbf{D}(x_0,k_0)+\frac{\partial \mathbf{D}}{\partial x}\cdot(x-x_0)+\frac{\partial \mathbf{D}}{\partial k}\cdot(k-k_0)+\cdots .$$

Explicitly

$$\mathbf{D}(x,k)=\begin{bmatrix} \frac{\partial D_a}{\partial x}\cdot(x-x_0)+\frac{\partial D_a}{\partial k}\cdot(k-k_0) & \eta' \\ \eta'^* & \frac{\partial D_b}{\partial x}\cdot(x-x_0)+\frac{\partial D_b}{\partial k}\cdot(k-k_0) \end{bmatrix},$$

where $D_a(x_0,k_0)=D_b(x_0,k_0)\equiv 0$, $\eta'\equiv\eta'(x_0,k_0)$, the derivatives are evaluated at (x_0,k_0), and we have neglected the higher-order terms. Before proceeding further we simplify $\mathbf{D}$ by shifting the origin in phase space to (x_0,k_0) (this results in multiplication of the wave field $\mathbf{Z}$ by an overall phase factor $\exp[ik_0\cdot(x-x_0)]$[3]),

$$\mathbf{D}(x,k)=\begin{bmatrix} \frac{\partial D_a}{\partial x}\cdot x+\frac{\partial D_a}{\partial k}\cdot k & \eta' \\ \eta'^* & \frac{\partial D_b}{\partial x}\cdot x+\frac{\partial D_b}{\partial k}\cdot k \end{bmatrix} . \tag{2.12}$$

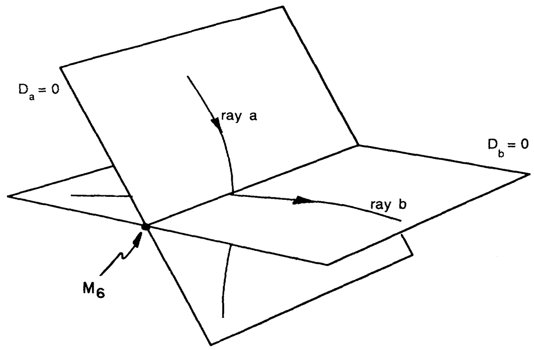

FIG. 1. The local phase space surrounding the conversion region. Shown are the dispersion manifolds $D_a=0$ and $D_b=0$. These seven-dimensional surfaces intersect transversely on the 6-dimensional subspace indicated as M_6. The rays of mode a lie in the surface $D_a=0$, the rays of mode b lie in the surface $D_b=0$. Ray a punctures the surface $D_b=0$ at a single point in M_6. This uniquely determines its counterpart, ray b.

The related pseudodifferential operator is

$$\mathcal{D}(\hat{x},\hat{k})=\begin{bmatrix} \frac{\partial D_a}{\partial x}\cdot\hat{x}+\frac{\partial D_a}{\partial k}\cdot\hat{k} & \eta' \\ \eta'^{*} & \frac{\partial D_b}{\partial x}\cdot\hat{x}+\frac{\partial D_b}{\partial k}\cdot\hat{k} \end{bmatrix} . \tag{2.13}$$

If we write

$$\mathcal{D}_a(\hat{x},\hat{k})\equiv\frac{\partial D_a}{\partial x}\cdot\hat{x}+\frac{\partial D_a}{\partial k}\cdot\hat{k},$$

$$\mathcal{D}_b(\hat{x},\hat{k})\equiv\frac{\partial D_b}{\partial x}\cdot\hat{x}+\frac{\partial D_b}{\partial k}\cdot\hat{k} ,$$

then using the commutation relations given in Sec. I and Eq. (2.6) it is straightforward to show that

$$[\mathcal{D}_a,\mathcal{D}_b]=i\{D_a,D_b\}_8$$

where the left-hand side is a relation between linear operators and the right-hand side a relation between phase-space functions.

D. Linear canonical transformations

We now carry out a linear canonical transformation to new phase-space coordinates which simplify the problem. We consider the classical canonical transformation here and the related operator transformation in Sec. III.

The goal is to find new coordinates such that $D_a(x,k)$ is proportional to the new momentum p_1 and $D_b(x,k)$ is proportional to the new position q_1. Following the notation in [12] we write $p_1=-B^{-1/2}D_a$ and $q_1=B^{-1/2}D_b$. Requiring $\{q_1,p_1\}_8=1$ leads to $B=\{D_a,D_b\}_8$. (If $\{D_a,D_b\}_8<0$ then B is defined with a minus sign.) Using Eq. (2.6) we find

$$\frac{dq_1}{d\sigma_a}=\{D_a,q_1\}_8=B^{1/2},\quad \frac{dp_1}{d\sigma_a}=\{D_a,p_1\}_8=0 .$$

Similarly

$$\frac{dq_1}{d\sigma_b}=\{D_b,q_1\}_8=0,\quad \frac{dp_1}{d\sigma_b}=\{D_b,p_1\}_8=B^{1/2} .$$

By Darboux's theorem [13] it is always possible to find the remaining canonical variables $(q_2,q_3,q_4,p_2,p_3,p_4)$. For completeness we show how to construct these coordinates in Appendix A. Note that, since they must commute with q_1 and p_1, this implies that they are all invariant along the rays:

$$\frac{dq_k}{d\sigma_n}=\frac{dp_k}{d\sigma_n}=0 \quad (k=2,3,4;n=a,b) .$$

We now introduce some notation that will make the ensuing algebra simpler. Organize the canonical coordinates into the following eight-dimensional vectors

$$\mathbf{z}\equiv\begin{bmatrix} x \\ k \end{bmatrix},\quad \mathbf{z}'\equiv\begin{bmatrix} q \\ p \end{bmatrix}$$

where x and k are the familiar four-vectors arranged in column form (similarly for q and p). The vectors $\mathbf{z}'$ and $\mathbf{z}$ are related via a linear transformation: $\mathbf{z}'=\mathbf{M}\mathbf{z}$, where $\mathbf{M}$ is an 8×8 symplectic matrix, i.e., it satisfies the identity

$$\mathbf{M}\mathbf{J}\tilde{\mathbf{M}}=\mathbf{J} \tag{2.14}$$

where the tilde denotes the transpose and the matrix $\mathbf{J}$ is defined as

$$\mathbf{J}\equiv\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Here $\mathbf{0}$ is the 4×4 zero matrix and $\mathbf{1}$ is the 4×4 identity matrix. From Eq. (2.14) we get $\det(\mathbf{M})=\pm 1$, therefore $\mathbf{M}^{-1}$ exists. Using $\mathbf{J}^2=-1$ it is easy to show that

$$\mathbf{M}^{-1}=-\mathbf{J}\tilde{\mathbf{M}}\mathbf{J} . \tag{2.15}$$

Following Ref. [2] we write the matrix $\mathbf{M}$ in the form

$$\mathbf{M}=\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} ,$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are 4×4 matrices. With the use of Eq. (2.15) it can be seen that

$$\mathbf{M}^{-1}=\begin{bmatrix} \tilde{\mathbf{D}} & -\tilde{\mathbf{B}} \\ -\mathbf{C} & \tilde{\mathbf{A}} \end{bmatrix} .$$

In the rest of the discussion we will assume that $\mathbf{B}$ is nonsingular. As discussed in [2] the case $\det\mathbf{B}=0$ is nongeneric and is associated with caustics. The symplectic condition, Eq. (2.14), implies

$$\mathbf{A}\tilde{\mathbf{D}}-\mathbf{B}\tilde{\mathbf{C}}=\mathbf{1},\quad \mathbf{A}\tilde{\mathbf{B}}=\mathbf{B}\tilde{\mathbf{A}},\quad \mathbf{C}\tilde{\mathbf{D}}=\mathbf{D}\tilde{\mathbf{C}} . \tag{2.16a}$$

The symplectic matrices form a group, therefore $\mathbf{M}^{-1}$ also satisfies Eq. (2.10) which implies [using Eq. (2.15)]:

$$\tilde{\mathbf{A}}\mathbf{D}-\tilde{\mathbf{C}}\mathbf{B}=\mathbf{1},\quad \tilde{\mathbf{A}}\mathbf{C}=\tilde{\mathbf{C}}\mathbf{A},\quad \tilde{\mathbf{B}}\mathbf{D}=\tilde{\mathbf{D}}\mathbf{B} . \tag{2.16b}$$

We consider the associated operator relations in the next section.

III. METAPLECTIC TRANSFORMATIONS

A. Analogy with rotation operators in quantum mechanics

In quantum mechanics one encounters the notion that there is an association between rotations in 3-space and a group of unitary operators in the Hilbert space, the rotation operators (see, for example, [14]). Specifically, suppose we perform a rotation of our spatial coordinates from $\mathbf{r}$ to $\mathbf{r}'$. This can be characterized by a 3×3 rotation matrix $\mathbf{R}$: $\mathbf{r}'=\mathbf{R}\mathbf{r}$. The rotation operator $\hat{\mathcal{R}}$ acts in the Hilbert space

$$\hat{\mathcal{R}}|\psi\rangle \equiv |\psi'\rangle \ .$$

It is possible to construct the rotation operator $\hat{\mathcal{R}}$ given the rotation matrix $\mathbf{R}$: $\hat{\mathcal{R}}=\hat{\mathcal{R}}(\mathbf{R})$.

There is an analogous relationship between a linear canonical transformation, associated with a symplectic matrix $\mathbf{M}$, and a unitary operator, $\hat{\mathcal{M}}$ which produces a change of representation in the Hilbert space:

$$\mathbf{M}\rightarrow\hat{\mathcal{M}}(\mathbf{M}) \ .$$

The operator $\hat{\mathcal{M}}$ is called a *metaplectic* operator [2]. The explicit representation of these operators is discussed in Sec. III B.

B. Relationship between symplectic matrices and metaplectic operator matrix elements

To start we introduce the following sets of eigenvectors: $\{|x\rangle\}$ are eigenvectors of the position operator $\hat{x}$ with eigenvalue x: $\hat{x}|x\rangle=x|x\rangle$. $\{|q\rangle\}$ are eigenvectors of the position operator $\hat{q}$ with eigenvalue q: $\hat{q}|q\rangle=q|q\rangle$. $\{|k\rangle\}$ are eigenvectors of the momentum operator $\hat{k}$ with eigenvalue k: $\hat{k}|k\rangle=k|k\rangle$. $\{|p\rangle\}$ are eigenvectors of the momentum operator $\hat{p}$ with eigenvalue p: $\hat{p}|p>=p|p\rangle$. Each of these four sets is complete and can be used as a basis. Here we shall simply state the matrix elements of the metaplectic operator and prove they have the desired properties. A detailed discussion can be found in [2] and references therein.

Suppose we wish to relate the operators $\hat{x}$ and $\hat{k}$ to a new set of operators $\hat{q}$ and $\hat{p}$ by the linear transformation

$$\hat{q}=\mathbf{A}\hat{x}+\mathbf{B}\hat{k}, \quad \hat{p}=\mathbf{C}\hat{x}+\mathbf{D}\hat{k} \ . \tag{3.1}$$

Here $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are real 4×4 matrices. Suppose we also wish the eight operators $\hat{q}$ and $\hat{p}$ to obey the canonical commutation relations

$$[\hat{q}^{\mu},\hat{p}_{\nu}]=i\delta^{\mu}_{\nu} \ .$$

Then, using Eq. (1.4) leads to the conditions required for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. A little algebra shows that these are precisely those given by Eqs. (2.16). The transformation $(\hat{x},\hat{k})\rightarrow(\hat{q},\hat{p})$ can also be carried out via a unitary transformation. Thus we wish the metaplectic operator to have the following properties:

$$\begin{aligned}\hat{\mathcal{M}}(\mathbf{M})\hat{x}\hat{\mathcal{M}}^{\dagger}(\mathbf{M})&=\hat{q}=\mathbf{A}\hat{x}+\mathbf{B}\hat{k} \ ,\\ \hat{\mathcal{M}}(\mathbf{M})\hat{k}\hat{\mathcal{M}}^{\dagger}(\mathbf{M})&=\hat{p}=\mathbf{C}\hat{x}+\mathbf{D}\hat{k} \ .\end{aligned} \tag{3.2}$$

The metaplectic operator maps the ket$|\psi\rangle$ into $|\psi'\rangle$:

$$|\psi'\rangle=\hat{\mathcal{M}}(\mathbf{M})|\psi\rangle \ .$$

$\hat{\mathcal{M}}$ transforms from the x representation to the q representation. Acting from the left with $\langle q|$ and inserting a complete set of x-space eigenfunctions we get

$$\langle q|\psi'\rangle=\int d^Nx\langle q|\hat{\mathcal{M}}(\mathbf{M})|x\rangle\langle x|\psi\rangle \ ,$$

or

$$\psi'(q)=\int d^Nx\ \mathcal{U}(q,x;\mathbf{M})\psi(x) \ . \tag{3.3}$$

Here N is the number of dimensions; in the present case $N=4$. The metaplectic matrix element $\mathcal{U}(q,x;\mathbf{M})$ can be constructed by choosing a representation (say the x representation) and solving Eq. (3.2) which, through the use of Eq. (3.3), becomes a partial differential equation for $\mathcal{U}(q,x;\mathbf{M})$. This is worked out in complete detail in [2], hence we merely quote the result

$$\mathcal{U}(q,x;\mathbf{M})=\frac{1}{(2\pi)^{N/2}|\det\mathbf{B}|^{1/2}}\times\exp\left[\frac{i}{2}(q\mathbf{B}^{-1}\mathbf{A}q-2q\mathbf{B}^{-1}x+x\mathbf{D}\mathbf{B}^{-1}x)\right] \ . \tag{3.4}$$

In order to simplify notation, in Eq. (3.4) the quadratic forms have been written as

$$q\mathbf{B}^{-1}x\equiv\sum_{j,k=1}^{N}q_j(B^{-1})_{jk}x_k \ ,$$

etc. There is an overall sign ambiguity (± 1) in Eq. (3.4) which we have ignored since it will not concern us here. The metaplectic operators preserve the multiplication rules of the symplectic group. Explicitly, given two symplectic matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ it is possible to show that

$$\hat{\mathcal{M}}(\mathbf{M}_1)\hat{\mathcal{M}}(\mathbf{M}_2)=\hat{\mathcal{M}}(\mathbf{M}_1\mathbf{M}_2) \ .$$

In the special case of $\mathbf{M}_2=\mathbf{M}_1^{-1}$ we have

$$\hat{\mathcal{M}}^{-1}(\mathbf{M})=\hat{\mathcal{M}}(\mathbf{M}^{-1}) \ .$$

Using Eqs. (3.4) and (2.16) we have

$$\mathcal{U}(x',q;\mathbf{M}^{-1})=\frac{1}{(2\pi)^{N/2}|\det\mathbf{B}|^{1/2}}\times\exp\left[-\frac{i}{2}(x'\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{D}}x'-2x\tilde{\mathbf{B}}^{-1}q+q\,\tilde{\mathbf{A}}\tilde{\mathbf{B}}^{-1}q)\right] \ , \tag{3.5}$$

Thus, in summary, to transform from the x to q representations we use Eq. (3.4), and from q to x representations Eq. (3.5).

1. Explicit calculation showing conversion of operators D_a and D_b to position and momentum operators

The purpose for the introduction of the metaplectic operator is to find a change of representation which simplifies the analysis of the mode-conversion problem. We seek to find a representation where the operators $\mathcal{D}_a$ and $\mathcal{D}_b$ are proportional to a conjugate pair of position and momentum operators $\hat{q}_1$ and $\hat{p}_1$. In this section we will show directly that the change of representation given by Eq. (3.5) converts the operator $\mathbf{A}\hat{x}+\mathbf{B}\hat{k}$ into a position operator in the new representation and $\mathbf{C}\hat{x}+\mathbf{D}\hat{k}$ into a momentum operator. Using Eq. (3.5) we write

$$\psi(x)=\int d^4q\ \mathcal{U}(x,q;\mathbf{M}^{-1})\psi'(q) \ ;$$

explicitly

$$\psi(x)=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\ e^{-i\phi(x,q)}\psi'(q) \ ,$$

where we define

$$\phi(x,q)\equiv\tfrac{1}{2}(q\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{D}}q-2q\tilde{\mathbf{B}}^{-1}x+x\,\tilde{\mathbf{A}}\tilde{\mathbf{B}}^{-1}x)$$
$$=\tfrac{1}{2}(q\mathbf{D}\mathbf{B}^{-1}q-2x\mathbf{B}^{-1}q+x\mathbf{B}^{-1}\mathbf{A}x) \ .$$

The equality of the two expressions for ϕ is due to Eqs. (2.16). Now consider the action of the x-space momentum operator $-i\partial_x$ on ϕ:

$$-i\partial_x\phi=-i[-\mathbf{B}^{-1}q+\mathbf{B}^{-1}\mathbf{A}x] \ .$$

Using this result we can calculate ψ_x:

$$\partial_x\psi(x)=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\times\int d^4q\ e^{-i\phi(x,q)}(-i\partial_x\phi)\psi'(q) \ ;$$

this leads to

$$[\mathbf{A}x-i\mathbf{B}\partial_x]\psi(x)=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\ e^{-i\phi(x,q)}[\mathbf{A}x-i\mathbf{B}(-i\partial_x\phi)]\psi'(q)$$
$$=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\ e^{-i\phi(x,q)}q\psi'(q) \ .$$

Thus we have the mapping $[\mathbf{A}x-i\mathbf{B}\partial_x]\to\hat{q}$ as desired. It remains to be shown that $[\mathbf{C}x-i\mathbf{D}\partial_x]\to -i\partial_q$:

$$[\mathbf{C}x-i\mathbf{D}\partial_x]\psi(x)=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\ e^{-i\phi(x,q)}[\mathbf{C}x-i\mathbf{D}(-i\partial_x\phi)]\psi'(q)$$
$$=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\ e^{-i\phi(x,q)}[\mathbf{C}x-\mathbf{D}(-\mathbf{B}^{-1}q+\mathbf{B}^{-1}\mathbf{A}x)]\psi'(q) \ .$$

We can use Eqs. (2.16) to show that the terms linear in x can be rewritten as

$$[\mathbf{C}-\mathbf{D}\mathbf{B}^{-1}\mathbf{A}]x=[\mathbf{C}\tilde{\mathbf{B}}-\mathbf{D}\,\tilde{\mathbf{A}}]\tilde{\mathbf{B}}^{-1}x=-\tilde{\mathbf{B}}^{-1}x \ .$$

Now using the symmetry of $\mathbf{D}\mathbf{B}^{-1}$ we can write

$$[\mathbf{C}x-i\mathbf{D}\partial_x]\psi(x)=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\ e^{-i\phi(x,q)}[-\tilde{\mathbf{B}}^{-1}x+\tilde{\mathbf{B}}^{-1}\mathbf{D}q]\psi'(q)$$
$$=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\left[i\frac{\partial}{\partial q}e^{-i\phi(x,q)}\right]\psi'(q) \ .$$

Integrating by parts we have, finally,

$$[\mathbf{C}x-i\mathbf{D}\partial_x]\psi(x)$$
$$=\frac{1}{(2\pi)^2|\det\mathbf{B}|^{1/2}}\int d^4q\ e^{-i\phi(x,q)}\left[-i\frac{\partial}{\partial q}\psi'(q)\right] .$$

As desired $[\mathbf{C}x-i\mathbf{D}\partial_x]\to -i\partial_q$.

2. Special example: The Fourier transform

Before returning to our discussion of the mode-conversion problem we wish to note that the Fourier transform is a special case of the metaplectic transform. In particular, consider the canonical transformation:

$$\begin{bmatrix} q \\ p \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ k \end{bmatrix} ,$$

with $\mathbf{A}=\mathbf{D}=\mathbf{0}$ and $\mathbf{B}=-\mathbf{C}=\mathbf{1}$. In this special case Eqs. (3.2) and (3.3) simplify to

$$\psi'(q)=\frac{1}{(2\pi)^2}\int d^4x\ e^{-iq\cdot x}\psi(x) \ ,$$

$$\psi(x)=\frac{1}{(2\pi)^2}\int d^4x\ e^{iq\cdot x}\psi'(q) \ .$$

The wider range of transformations available using metaplectic techniques can be put to good advantage in the mode-conversion problem, as we discuss in the next section.

IV. APPLICATIONS OF METAPLECTIC TRANSFORMATIONS TO LINEAR MODE CONVERSION

A. Choosing representations

Consider Eq. (2.1) once again. Under a metaplectic transformation the pseudodifferential operator $\mathcal{D}(\hat{x},\hat{k})$ associated with the symbol $\mathbf{D}(x,k)$ transforms in the simple manner:

$$\hat{\mathcal{M}}(\mathbf{M})\mathcal{D}(\hat{x},\hat{k})\hat{\mathcal{M}}^{\dagger}(M)=\mathcal{D}(\hat{q},\hat{p}) .$$

We consider the transformation properties in more detail in Appendix B. Here we shall simply state that the order of operations is irrelevant: one can associate the symbol with a pseudodifferential operator $\mathbf{D}(x,k)\rightarrow\mathcal{D}(\hat{x},\hat{k})$ and then perform a metaplectic transformation $\hat{\mathcal{M}}(\mathbf{M})$ to arrive at $\mathcal{D}(\hat{q},\hat{p})$, or one can perform a linear *classical* canonical transformation on the symbol $\mathbf{D}(x,k)\rightarrow\mathbf{D}(q,p)$ and then associate the new symbol to an operator $\mathbf{D}(q,p)\rightarrow D(\hat{q},\hat{p})$. One arrives at the same result either way, as required for the procedure to be sensible.

We expand Eq. (2.1) about the conversion point (x_0,k_0) as in Eq. (2.13). Now multiply by $B^{-1/2}$ (recall $B\equiv\{D_a,D_b\}_8$) and define $\eta\equiv B^{-1/2}\eta'(x_0,k_0)$. This transforms Eq. (2.1) into

$$\begin{bmatrix} B^{-1/2}D_a(x,k) & \eta \\ \eta^* & B^{-1/2}D_b(x,k)\end{bmatrix}\begin{bmatrix} Z_a \\ Z_b\end{bmatrix}=0 ,$$

where

$$D_a(x,k)\equiv\frac{\partial D_a}{\partial x}\cdot x+\frac{\partial D_a}{\partial k}\cdot k,$$
$$D_b(x,k)\equiv\frac{\partial D_b}{\partial x}\cdot x+\frac{\partial D_b}{\partial k}\cdot k .$$

The associated operator relation is

$$\begin{bmatrix} B^{-1/2}\mathcal{D}_a(\hat{x},\hat{k}) & \eta \\ \eta^* & B^{-1/2}\mathcal{D}_b(\hat{x},\hat{k})\end{bmatrix}\begin{bmatrix} Z_a \\ Z_b\end{bmatrix}=0 , \quad (4.1)$$

where $\mathcal{D}_a$ and $\mathcal{D}_b$ are defined in Eq. (2.13). As mentioned previously (and discussed in Appendix A) it is possible to construct a linear canonical transformation which takes $B^{-1/2}\mathcal{D}_a$ to $-p_1$ and $B^{-1/2}\mathcal{D}_b$ to q_1. This is done via a symplectic matrix $\mathbf{M}$. The associated metaplectic transformation, when applied to Eq. (4.1), has the following effect:

$$\mathcal{M}(\mathbf{M})\begin{bmatrix} B^{-1/2}\mathcal{D}_a(\hat{x},\hat{k}) & \eta \\ \eta^* & B^{-1/2}\mathcal{D}_b(\hat{x},\hat{k})\end{bmatrix}$$
$$\times\mathcal{M}^{\dagger}(\mathbf{M})\mathcal{M}(\mathbf{M})\begin{bmatrix} Z_a \\ Z_b\end{bmatrix}=\begin{bmatrix} -\hat{p}_1 & \eta \\ \eta' & \hat{q}_1\end{bmatrix}\begin{bmatrix} Z_a' \\ Z_b'\end{bmatrix}=0 , \quad (4.2)$$

In the q representation (we drop the primes on the wave functions for convenience) this becomes

$$\begin{bmatrix} i\dfrac{\partial}{\partial q_1} & \eta \\ \eta^* & q_1\end{bmatrix}\begin{bmatrix} Z_a(q) \\ Z_b(q)\end{bmatrix}=0 \quad (4.3)$$

while in the p representation we have

$$\begin{bmatrix} -p_1 & \eta \\ \eta^* & i\dfrac{\partial}{\partial p_1}\end{bmatrix}\begin{bmatrix} Z_a(p) \\ Z_b(p)\end{bmatrix}=0 . \quad (4.4)$$

Because the variables q and p are canonically conjugate their representations are connected via Fourier transforms

$$\mathbf{Z}(p)=\frac{1}{(2\pi)^2}\int d^4q\, e^{-ip\cdot q}\mathbf{Z}(q) . \quad (4.5)$$

Notice that both Eqs. (4.3) and (4.4) are first-order *ordinary* differential equations in the variables q_1 and p_1, respectively. The dependence of the wave function on the other variables $[(q_2,q_3,q_4)$ in the q representation and (p_2,p_3,p_4) in the p representation$]$ is related to the boundary conditions.

B. Solutions in the conversion region

The reader may find it helpful to refer to Fig. 2 during the following discussion. To simplify the algebra we introduce the following notation:

$$q=(q_1,q_2,q_3,q_4)\equiv(q_1,\mathbf{q}),\quad p=(p_1,p_2,p_3,p_4)\equiv(p_1,\mathbf{p}) .$$

Consider Eq. (4.3). Eliminating $Z_b(q)$ we find that $Z_a(q)$ satisfies the following ordinary differential equation:

$$i\frac{\partial Z_a}{\partial q_1}=\frac{|\eta|^2}{q_1}Z_a . \quad (4.6)$$

Notice the singularity at $q_1=0$ where the conversion occurs. We split the real q_1 axis into positive and negative halves and treat each half separately. The solution of

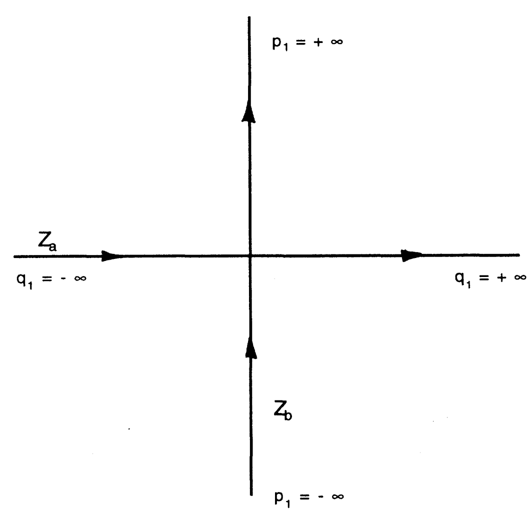

FIG. 2. Schematic of the conversion process in the q_1-p_1 plane. Away from the conversion region, mode a is confined to the surface $p_1=0$ and mode b is confined to the surface $q_1=0$. Mode a enters from $q_1=-\infty$ and leaves at $q_1=+\infty$. Mode b enters from $p_1=-\infty$ and leaves at $p_1=+\infty$.

Eq. (4.6) is

$$Z_a(q)=\begin{cases}\alpha_+(\mathbf{q})q_1^{-i|\eta|^2}, & q_1>0\\ \alpha_-(\mathbf{q})|q_1|^{-i|\eta|^2}, & q_1<0\ .\end{cases} \quad (4.7a)$$

From Eq. (4.3) we see that

$$Z_b(q)=-\frac{\eta^*}{q_1}Z_a(q)\ ,$$

and therefore

$$Z_b(q)=\begin{cases}-\eta^*\alpha_+(\mathbf{q})q_1^{-i|\eta|^2-1}, & q_1>0\\ \eta^*\alpha_-(\mathbf{q})|q_1|^{-i|\eta|^2-1}, & q_1<0\ .\end{cases} \quad (4.7b)$$

The p-space representation of $\mathbf{Z}$ can be found by Fourier transforming according to Eq. (4.5). The results are (details are in Appendix C)

$$Z_a(p)=\begin{cases}\dfrac{p_1^{i|\eta|^2-1}\Gamma(-i|\eta|^2+1)}{(2\pi)^{1/2}}\left[-i\alpha_-(\mathbf{p})e^{\pi|\eta|^2/2}+i\alpha_+(\mathbf{p})e^{-\pi|\eta|^2/2}\right], & p_1>0\\ \dfrac{|p_1|^{i|\eta|^2-1}\Gamma(-i|\eta|^2+1)}{(2\pi)^{1/2}}\left[i\alpha_-(\mathbf{p})e^{-\pi|\eta|^2/2}-i\alpha_+(\mathbf{p})e^{\pi|\eta|^2/2}\right], & p_1<0\ ,\end{cases} \quad (4.8a)$$

$$Z_b(p)=\begin{cases}\dfrac{\eta^* p_1^{i|\eta|^2}\Gamma(-i|\eta|^2)}{(2\pi)^{1/2}}\left[\alpha_-(\mathbf{p})e^{\pi|\eta|^2/2}-\alpha_+(\mathbf{p})e^{-\pi|\eta|^2/2}\right], & p_1>0\\ \dfrac{\eta^*|p_1|^{i|\eta|^2}\Gamma(-i|\eta|^2)}{(2\pi)^{1/2}}\left[\alpha_-(\mathbf{p})e^{-\pi|\eta|^2/2}-\alpha_+(\mathbf{p})e^{\pi|\eta|^2/2}\right], & p_1<0\ .\end{cases} \quad (4.8b)$$

In order to construct the S matrix we compare the incoming and outgoing field amplitudes, defined as

$$\begin{aligned}\bar{Z}_a(\pm q_1)&\equiv|q_1|^{i|\eta|^2}Z_a(\pm q_1)\ ,\\ \bar{Z}_b(\pm p_1)&\equiv|p_1|^{-i|\eta|^2}Z_b(\pm p_1)\ .\end{aligned} \quad (4.9)$$

For this comparison to be physically meaningful the boundary conditions must be evaluated in the same representation. We Fourier transform Eq. (4.8b) with respect to $\mathbf{p}$ (but not with respect to p_1), converting $\alpha_\pm(\mathbf{p})$ to $\alpha_\pm(\mathbf{q})$. Using Eqs. (4.7)–(4.9) we finally arrive at

$$\begin{bmatrix}\bar{Z}_a(+q_1)\\ \bar{Z}_b(+p_1)\end{bmatrix}=\begin{bmatrix}\tau & -\beta\\ \beta^* & \tau\end{bmatrix}\begin{bmatrix}\bar{Z}_a(-q_1)\\ \bar{Z}_b(-p_1)\end{bmatrix}\ , \quad (4.10)$$

where

$$\tau(\eta)\equiv\exp(-\pi|\eta|^2),\quad \beta(\eta)\equiv\frac{(2\pi\tau)^{1/2}}{\eta\Gamma(-i|\eta|^2)}\ .$$

Construction of Eq. (4.10) requires use of the identity [Eq. (6.1.29) of Ref. [15]]

$$\Gamma(i|\eta|^2)\Gamma(-i|\eta|^2)=|\Gamma(i|\eta|^2)|^2=\frac{\pi}{|\eta|^2\sinh(\pi|\eta|^2)}\ . \quad (4.11)$$

The S matrix

$$S(\eta)\equiv\begin{bmatrix}\tau & -\beta\\ \beta^* & \tau\end{bmatrix} \quad (4.12)$$

can easily be shown to be unitary using Eq. (4.11). Recall that $\eta=\eta(x_0,k_0)$ in (x,k) coordinates and $\eta=\eta(q_1=0,p_1=0,\mathbf{q}_0,\mathbf{p}_0)$ in (q,p) coordinates. Therefore $\mathbf{S}(\eta)=\mathbf{S}(\mathbf{q}_0,\mathbf{p}_0)$.

Equations (4.7)–(4.12) completely characterize the solutions in the mode-conversion region. Given $Z(q)$ we can the find $Z(x)$ by performing a metaplectic transformation Eq. (3.5). Formally

$$\mathbf{Z}(x)=\frac{1}{(2\pi)^{N/2}|\det\mathbf{B}|^{1/2}}\int d^Nq\,\exp\left[\frac{-i}{2}(q\mathbf{D}\mathbf{B}^{-1}q-2x\mathbf{B}^{-1}q+x\mathbf{B}^{-1}\mathbf{A}x)\right]\alpha(\mathbf{q})q_1^{-i|\eta|^2}\ ,$$

where $\alpha(\mathbf{q})=\alpha_+(\mathbf{q})$ $[\alpha_-(\mathbf{q})]$ for $q_1>0$ $(q_1<0)$. We have not pursued this path; instead we use the solutions (4.7) to construct the Wigner tensor in the mode-conversion region and examine its asymptotic behavior as we leave the immediate vicinity of the conversion point.

We now consider the Wigner tensor in the conversion region.

C. The Wigner tensor

Recall that the Wigner tensor is the 2×2 matrix defined in Eq. (2.5). Because the Wigner function is a scalar under metaplectic transformations (see Appendix D) we can evaluate it in whatever representation is most convenient. We choose, of course, the (q,p) representation. Here we compute the Wigner function associated with mode a, W_{aa}. The Wigner function for mode b, W_{bb} (and the off-diagonal cross terms W_{ab} and W_{ba}) can be computed in similar fashion. W_{aa} is defined in the standard fashion:

$$\mathsf{W}_{aa}(q,p)=\int d^4s\,e^{-ip\cdot s}Z_a(q+\tfrac{1}{2}s)Z_a^*(q-\tfrac{1}{2}s)\ , \quad (4.13)$$

where $p\cdot s$ is the standard four-vector inner product: $-p_1 s_1+\mathbf{p}\cdot\mathbf{s}$. Because of the discontinuity at $q_1=0$ we break the s_1 integral into three parts (see Fig. 3). The value of $\operatorname{sgn}(q_1\pm s_1/2)$ determines whether one must use α_+ or α_- in each region.

Region I:

$$q_1+\frac{s_1}{2}=q_1-\frac{|s_1|}{2}<0,\quad q_1-\frac{s_1}{2}=q_1+\frac{|s_1|}{2}>0 .$$

This implies that

$$\begin{aligned} Z_a\left(q+\frac{s}{2}\right)&=\alpha_-\left(\mathbf{q}+\frac{\mathbf{s}}{2}\right)\left|q_1+\frac{s_1}{2}\right|^{-i|\eta|^2},\\ Z_a^*\left(q-\frac{s}{2}\right)&=\alpha_+^*\left(\mathbf{q}-\frac{\mathbf{s}}{2}\right)\left(q_1-\frac{s_1}{2}\right)^{i|\eta|^2}. \end{aligned} \tag{4.14}$$

Notice that this result is independent of $\operatorname{sgn}(q_1)$.

Region II. In this region we have $\operatorname{sgn}(q_1\pm s_1/2)=\operatorname{sgn}(q_1)$. This implies

$$Z_a\left(q+\frac{s}{2}\right)=\begin{cases}\alpha_+\left(\mathbf{q}+\frac{\mathbf{s}}{2}\right)\left(q_1+\frac{s_1}{2}\right)^{-i|\eta|^2}, & q_1>0\\ \alpha_-\left(\mathbf{q}+\frac{\mathbf{s}}{2}\right)\left|q_1+\frac{s_1}{2}\right|^{-i|\eta|^2}, & q_1<0,\end{cases} \tag{4.15a}$$

$$Z_a^*\left(q-\frac{s}{2}\right)=\begin{cases}\alpha_+^*\left(\mathbf{q}-\frac{\mathbf{s}}{2}\right)\left(q_1-\frac{s_1}{2}\right)^{i|\eta|^2}, & q_1>0\\ \alpha_-^*\left(\mathbf{q}-\frac{\mathbf{s}}{2}\right)\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2}, & q_1<0 .\end{cases} \tag{4.15b}$$

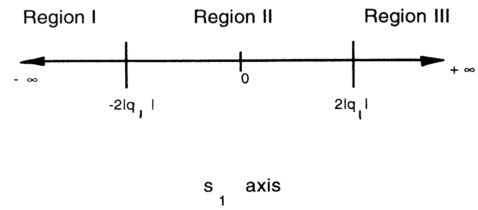

FIG. 3. A diagram showing the partition of the real s_1 line into the three regions defined in the text.

Region III: In this region we have

$$q_1+\frac{s_1}{2}>0,\quad q_1-\frac{s_1}{2}<0 .$$

The resulting wave functions are

$$\begin{aligned} Z_a\left(q+\frac{s}{2}\right)&=\alpha_+\left(\mathbf{q}+\frac{\mathbf{s}}{2}\right)\left(q_1+\frac{s_1}{2}\right)^{-i|\eta|^2},\\ Z_a^*\left(q-\frac{s}{2}\right)&=\alpha_-\left(\mathbf{q}-\frac{\mathbf{s}}{2}\right)\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2}. \end{aligned} \tag{4.16}$$

Notice that this result is independent of $\operatorname{sgn}(q_1)$, as in region I.

We can now use these results to evaluate the Wigner function $\mathsf{W}_{aa}(q,p)$. Consider first the incoming Wigner function, i.e., we fix $q_1<0$. Using Eqs. (4.14)–(4.16) we find

$$\begin{aligned} \mathsf{W}_{aa}(q,p)=&\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_-\left(\mathbf{q}+\frac{\mathbf{s}}{2}\right)\alpha_+^*\left(\mathbf{q}-\frac{\mathbf{s}}{2}\right)\int_{-\infty}^{-2|q_1|}ds_1 e^{ip_1s_1}\left|q_1+\frac{s_1}{2}\right|^{-i|\eta|^2}\left(q_1-\frac{s_1}{2}\right)^{i|\eta|^2}\\ &+\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_-\left(\mathbf{q}+\frac{\mathbf{s}}{2}\right)\alpha_-^*\left(\mathbf{q}-\frac{\mathbf{s}}{2}\right)\int_{-2|q_1|}^{2|q_1|}ds_1 e^{ip_1s_1}\left|q_1+\frac{s_1}{2}\right|^{-i|\eta|^2}\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2}\\ &+\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_+\left(\mathbf{q}+\frac{\mathbf{s}}{2}\right)\alpha_-^*\left(\mathbf{q}-\frac{\mathbf{s}}{2}\right)\int_{2|q_1|}^{\infty}ds_1 e^{ip_1s_1}\left(q_1+\frac{s_1}{2}\right)^{-i|\eta|^2}\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2} . \end{aligned} \tag{4.17}$$

Here the first term is the contribution from region I, the second from region II, and the third from region III. Notice that the contributions from regions I and III are not causal: they involve both α_- (initial conditions) and α_+ (final conditions). The contribution from region II, however, involves only the initial conditions. This disturbing appearance of mixed causal behavior is due to the nonlocal nature of the Wigner function. It is possible to show, however, that the relevant physical quantities are causal. For example, as we shall show in a moment, W_{aa} becomes confined to the surface $p_1=0$ asymptotically as we leave the conversion region. Away from the conversion region physical quantities associated with mode a, like the energy or action, are therefore distributions on the surface $p_1=0$ and we recover them from W_{aa} by integrating across this surface [11]:

$$J_a(q,\mathbf{p})\equiv\int dp_1\mathsf{W}_{aa}(q,p) .$$

(Please note that the quantity called the action in [11] actually has dimensions of energy density. This can be rectified by modifying some of the physical arguments in [11] without changing the mathematical logic of how to extract physical quantities from the Wigner tensor, which is all that concerns us here.) Also, notice that the quantity $J_a(q,\mathbf{p})$ is *not* invariant with respect to arbitrary linear canonical transformations, but only those involving the $(\mathbf{q},\mathbf{p})$ subspace.

Performing an integration in p_1 upon Eq. (4.17) and then exchanging the order of integrations in p_1 and s_1 leads to

$$J_a(q,\mathbf{p})=\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_-\left[\mathbf{q}+\frac{\mathbf{s}}{2}\right]\alpha_+^*\left[\mathbf{q}-\frac{\mathbf{s}}{2}\right]\int_{-\infty}^{-2|q_1|}ds_1 2\pi\delta(s_1)\left[q_1+\frac{s_1}{2}\right]^{-i|\eta|^2}\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2}$$
$$+\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_-\left[\mathbf{q}+\frac{\mathbf{s}}{2}\right]\alpha_-^*\left[\mathbf{q}-\frac{\mathbf{s}}{2}\right]\int_{-2|q_1|}^{2|q_1|}ds_1 2\pi\delta(s_1)\left|q_1+\frac{s_1}{2}\right|^{-i|\eta|^2}\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2}$$
$$+\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_+\left[\mathbf{q}+\frac{\mathbf{s}}{2}\right]\alpha_-^*\left[\mathbf{q}-\frac{\mathbf{s}}{2}\right]\int_{2|q_1|}^{\infty}ds_1 2\pi\delta(s_1)\left[q_1+\frac{s_1}{2}\right]^{-i|\eta|^2}\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2} .$$

The integrations in s_1 can now be performed and we find that the noncausal contributions from regions I and III vanish since the δ function is identically zero in regions I and III. We are left only with the contribution from region II:

$$J_a(q,\mathbf{p})=2\pi\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_-\left[\mathbf{q}+\frac{\mathbf{s}}{2}\right]\alpha_-^*\left[\mathbf{q}-\frac{\mathbf{s}}{2}\right]$$
$$\equiv 2\pi \mathsf{W}_{aa}^{(-)}(\mathbf{q},\mathbf{p}),\quad q_1<0 . \tag{4.18a}$$

Here $\mathsf{W}_{aa}^{(-)}$ is the Wigner function of the initial conditions. Notice that J_a is independent of q_1. If we repeat the calculation for $q_1>0$ we find

$$J_a(q,\mathbf{p})=2\pi\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_+\left[\mathbf{q}+\frac{s}{2}\right]\alpha_+^*\left[\mathbf{q}-\frac{\mathbf{s}}{2}\right]$$
$$\equiv 2\pi \mathsf{W}_{aa}^{(+)}(\mathbf{q},\mathbf{p}),\quad q_1>0 . \tag{4.18b}$$

The functions α_+ and α_- can be related using the S matrix, Eq. (4.12).

Let us examine the Wigner function, Eq. (4.17), in more detail. In particular, let us consider the causal contribution (call it W_{aa}^c)

$$\mathsf{W}_{aa}^c(q,p)\equiv\int d^3\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\alpha_-\left[\mathbf{q}+\frac{\mathbf{s}}{2}\right]\alpha_-^*\left[\mathbf{q}-\frac{\mathbf{s}}{2}\right]$$
$$\times\int_{-2|q_1|}^{2|q_1|}ds_1 e^{ip_1s_1}\left|q_1+\frac{s_1}{2}\right|^{-i|\eta|^2}$$
$$\times\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2} \tag{4.19}$$

The s_1 integrand involves the function

$$\left|q_1+\frac{s_1}{2}\right|^{-i|\eta|^2}\left|q_1-\frac{s_1}{2}\right|^{i|\eta|^2}$$
$$=\left|\frac{q_1-\frac{s_1}{2}}{q_1+\frac{s_1}{2}}\right|^{i|\eta|^2}=\left|\frac{1-\frac{s_1}{2q_1}}{1+\frac{s_1}{2q_1}}\right|^{i|\eta|^2}$$
$$=\left[\frac{1-\frac{s_1}{2q_1}}{1+\frac{s_1}{2q_1}}\right]^{i|\eta|^2} .$$

In the last step we have made use of the fact that, between the limits of integration, we have $|s_1/2q_1|<1$. Therefore we can expand this expression as a power series in s_1:

$$\left[\frac{1-\frac{s_1}{2q_1}}{1+\frac{s_1}{2q_1}}\right]^{i|\eta|^2}=\sum_{n=0}^{\infty}a_n\left[\frac{s_1}{2q_1}\right]^n$$
$$=1-i|\eta|^2\frac{s_1}{q_1}+O\left[\left[\frac{s_1}{q_1}\right]^2\right] .$$

We have explicitly evaluated only the first two terms in the expansion. The function

$$\left[\frac{1-x}{1+x}\right]^{i|\eta|^2}$$

is singular at $x=\pm1$ and the expansion breaks down. However, it oscillates rapidly near ±1 and, therefore, the end points of the integration should give a negligible contribution to the integral. This justifies integrating the Taylor expansion term by term, leading to integrals of the form

$$\int_{-2|q_1|}^{2|q_1|}ds_1 e^{ip_1s_1}s_1^n=\left[-i\frac{\partial}{\partial p_1}\right]^n\int_{-2|q_1|}^{2|q_1|}ds_1 e^{ip_1s_1}$$
$$=2\left[-i\frac{\partial}{\partial p_1}\right]^n\left[\frac{\sin(2p_1|q_1|)}{p_1}\right] . \tag{4.20}$$

Here it is important to notice that all of the terms with $n\neq0$ can be expressed as a derivative in p_1. This explains why, when we compute the action using Eqs. (4.18), there are no contributions from the higher-order terms. Only the $n=0$ term survives the integration, leading to the result quoted in Eqs. (4.18) even when q_1 is close to the origin. In the limit that $|q_1|$ tends to infinity we have

$$\lim_{|q_1|\to\infty}\left[2\frac{\sin(2p_1|q_1|)}{p_1}\right]=2\pi\delta(p_1) .$$

This shows that, far from the mode-conversion region near $q_1=0$, the Wigner function becomes confined to the dispersion manifold given by $p_1=0$. For finite q_1, however, the Wigner function has nonzero support away from $p_1=0$.

In summary, in this paper we have given a detailed discussion of the application of metaplectic transformations to linear mode conversion. After introducing the necessary background material, we have examined the solutions of the wave equation in the conversion region in detail. This leads to the fact that in the mode-conversion region the Wigner function has mixed causality (i.e., it mixes incoming and outgoing data). This is due to the nonlocal nature of the Wigner function and not to any particular choice of representation. This acausal behavior does not affect the final physical result. The Wigner function is a very complicated distribution on the phase space and to compute physical quantities, such as action or energy densities, one must do projections along directions normal to the relevant dispersion surfaces. Upon performing such projections only the causal part of the Wigner function remains. This causal part was also shown to become asymptotically confined to the relevant dispersion manifolds, as expected from the prior WKB analysis.

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APPENDIX A: CONSTRUCTION OF THE COMPLETE SET OF CANONICAL VARIABLES

We now introduce some notation that will make the ensuing algebra simpler. Organize the canonical coordinates into the following eight-dimensional vectors

$$\mathbf{z} \equiv \begin{bmatrix} x \\ k \end{bmatrix}, \quad \mathbf{z}' \equiv \begin{bmatrix} q \\ p \end{bmatrix} \tag{A1}$$

where x and k are the familiar four-vectors arranged in column form (similarly for q and p). The linearized dispersion functions can be written in the compact form

$$D_a(x,k) = \mathbf{d}_a \cdot \mathbf{z}, \quad D_b(x,k) = \mathbf{d}_b \cdot \mathbf{z}$$

where the dot is the standard Euclidean dot product and the vectors $\mathbf{d}_a$ and $\mathbf{d}_b$ are defined as

$$\mathbf{d}_a \equiv B^{-1/2} \begin{bmatrix} \dfrac{\partial D_a}{\partial x} \\ \dfrac{\partial D_a}{\partial k} \end{bmatrix}, \quad \mathbf{d}_b \equiv B^{-1/2} \begin{bmatrix} \dfrac{\partial D_b}{\partial x} \\ \dfrac{\partial D_b}{\partial k} \end{bmatrix}.$$

Consider now two general linear functions on the phase space

$$f(z) \equiv \nabla f \cdot \mathbf{z}, \quad g(z) \equiv \nabla g \cdot \mathbf{z} .$$

The Poisson bracket between linear functions on the phase-space induces an antisymmetric bilinear product between vectors:

$$\{f,g\}_8 = \frac{\partial f}{\partial z_m} J_{mn} \frac{\partial g}{\partial z_n} = \nabla f \cdot \mathbf{J} \cdot \nabla g . \tag{A2}$$

To construct a complete set of canonical variables we use a symplectic version of the Gram-Schmidt orthogonalization process. We first assume that we have $2N$ independent linear functions on the phase space, which we call λ_m and σ_m for $m = 1,2,\ldots,N$. (Equivalently, we could state this assumption as the existence of $2N$ independent vectors.) We choose $\lambda_1 = D_a$ and $\sigma_1 = D_b$. Then

$$q_1 = B^{-1/2}\lambda_1, \quad p_1 = -B^{-1/2}\sigma_1$$

as before. We now construct the other coordinates by induction starting with (q_2,p_2). Define (q_2,p_2) as follows:

$$\begin{aligned} q_2 &\equiv \alpha_2(\lambda_2 + \{\lambda_2,q_1\}_8 p_1 - \{\lambda_2,p_1\}_8 q_1) , \\ p_2 &\equiv \alpha_2(\sigma_2 + \{\sigma_2,q_1\}_8 p_1 - \{\sigma_2,p_1\}_8 q_1) . \end{aligned} \tag{A3}$$

Straightforward calculation shows that

$$\{q_1,q_2\}_8 = \{q_1,p_2\}_8 = \{p_1,q_2\}_8 = \{p_1,p_2\}_8 = 0 ,$$

as desired. The constant α_2 is determined by the requirement $\{q_2,p_2\}_8 = 1$. The geometric meaning of definitions (A3) can be made a little less mysterious if we introduce the following notation: define the two-dimensional vector $\mathbf{z}_{(1)}$ as the restriction of $\mathbf{z}$ to the (q_1,p_1) subspace, i.e.,

$$\mathbf{z}_{(1)} \equiv \begin{bmatrix} q_1 \\ p_1 \end{bmatrix} .$$

Define the 2×2 matrix $J^{(2)}$ as

$$\mathbf{J}^{(2)} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Then (A3) can be written more compactly as

$$\begin{aligned} q_2 &\equiv \alpha_2(\lambda_2 + \{\lambda_2,\mathbf{z}_{(1)}\}_8 \mathbf{J}^{(2)} \mathbf{z}_{(1)}) , \\ p_2 &\equiv \alpha_2(\sigma_2 + \{\sigma_2,\mathbf{z}_{(1)}\}_8 \mathbf{J}^{(2)} \mathbf{z}_{(1)}) . \end{aligned} \tag{A4}$$

Taking the Poisson bracket of q_2 and p_2 now leads to

$$\{q_2,p_2\}_8 = \alpha_2^2[\{\lambda_2,\sigma_2\}_8 + \{\lambda_2,\mathbf{z}_{(1)}\}_8 \mathbf{J}^{(2)} \{\mathbf{z}_{(1)},\sigma_2\}_8] . \tag{A5}$$

The Poisson bracket $\{\ ,\ \}_g$ is invariant under canonical transformations; therefore we can evaluate the Poisson brackets in Eqs. (A4) and (A5) in terms of the new coordinates

$$\{\mathbf{z}_{(1)},\lambda_2\}_8 = \begin{bmatrix} \dfrac{\partial \lambda_2}{\partial p_1} \\ -\dfrac{\partial \lambda_2}{\partial q_1} \end{bmatrix} ;$$

and after a little algebra we are led to

$$\begin{aligned} \{q_2,p_2\}_8 &= \alpha_2^2[\{\lambda_2,\sigma_2\}_8 - \{\lambda_2,\sigma_2\}_{(1)}] \\ &\equiv \alpha_2^2\{\lambda_2,\sigma_2\}_{(1)\perp} , \end{aligned} \tag{A6}$$

where $\{\lambda_2,\sigma_2\}_{(1)}$ is the Poisson bracket restricted to the (q_1,p_1) subspace

$$\{f,g\}_{(1)}\equiv \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1}-\frac{\partial g}{\partial q_1}\frac{\partial f}{\partial p_1}$$

and $\{\lambda_2,\sigma_2\}_{(1)\perp}$ is the Poisson bracket restricted to the perpendicular complement of the (q_1,p_1) subspace.

The variables (q_3,p_3) are given by

$$q_3\equiv\alpha_3(\lambda_3+\{\lambda_3,\mathbf{z}_{(1,2)}\}_8\mathbf{J}^{(4)}\mathbf{z}_{(1,2)}) ,$$

$$p_3\equiv\alpha_3(\sigma_3+\{\sigma_3,\mathbf{z}_{(1,2)}\}_8\mathbf{J}^{(4)}\mathbf{z}_{(1,2)}) .$$

Here $\mathbf{z}_{(1,2)}$ is the restriction of $\mathbf{z}$ to the four-dimensional subspace spanned by (q_1,p_1,q_2,p_2) and $\mathbf{J}^{(4)}$ is the 4×4 antisymmetric matrix

$$\mathbf{J}^{(4)}\equiv\begin{bmatrix}\mathbf{0} & \mathbf{1}\\ -\mathbf{1} & \mathbf{0}\end{bmatrix}$$

where $\mathbf{0}$ and $\mathbf{1}$ are the 2×2 zero and identity matrices, respectively. The constant α_3 is fixed by requiring $\{q_3,p_3\}=1$.

The induction procedure should now be clear. In this manner one can construct the complete set of canonical variables.

APPENDIX B: TRANSFORMATION PROPERTIES OF SYMBOLS AND PSEUDODIFFERENTIAL OPERATORS

In the following discussion we ignore all convergence issues as they are outside the scope of our discussion. The interested reader is referred to the paper by Hörmander [9]. Consider the procedure discussed in Sec. I for relating symbols to pseudodifferential operators. We first construct the Fourier representation of the symbol

$$\mathbf{D}(x,k)\equiv\int d^4\sigma\, d^4\tau\, e^{i(\sigma\cdot x+\tau\cdot k)}\overline{\mathbf{D}}(\sigma,\tau) .$$

The pseudodifferential operator associated with $\mathbf{D}(x,k)$ is defined as

$$\mathcal{D}(\hat{x},\hat{k})=\int d^4\sigma\, d^4\tau\overline{\mathbf{D}}(\sigma,\tau)\exp[i(\sigma\cdot\hat{x}+\tau\cdot\hat{k})] . \tag{B2}$$

Consider the effect of a linear canonical transformation on these relations. The algebra is simplified and the logic clarified substantially if we introduce some new notation. As in Appendix A we write

$$\mathbf{z}\equiv\begin{bmatrix}x\\ k\end{bmatrix},\quad \mathbf{z}'\equiv\begin{bmatrix}q\\ p\end{bmatrix}=\mathbf{Mz}$$

and now introduce

$$\zeta\equiv\begin{bmatrix}\sigma\\ \tau\end{bmatrix} .$$

The vector $\mathbf{z}$ is an element of the vector space V. The vector ζ is an element of the dual space V^*. The inner product on V is defined as the standard Euclidean dot product

$$\langle\zeta,z\rangle\equiv\sigma\cdot x+\tau\cdot k=\zeta\cdot z .$$

Let us focus for the moment on linear functions on the phase space, i.e., functions of the form

$$f(\mathbf{z})\equiv\mathbf{a}\cdot\mathbf{z},\quad a\in V^* .$$

Consider the canonical transformation $\mathbf{z}\to\mathbf{z}'=\mathbf{Mz}$. This transforms f to f':

$$f(z')=\mathbf{a}\cdot\mathbf{z}'=\mathbf{a}\cdot(\mathbf{Mz})=(\tilde{\mathbf{M}}\mathbf{a})\cdot\mathbf{z}=f'(\mathbf{z}) . \tag{B3}$$

Thus the canonical transformation $\mathbf{z}\to\mathbf{z}'=\mathbf{Mz}$ on V has the same effect as the transformation $a\to a'=\mathbf{M}a$ on V^*. The transformation from $\mathbf{a}$ to $\mathbf{a}'$ is also canonical. This can be shown by first using the fact that

$$\tilde{\mathbf{M}}=-\mathbf{JM}^{-1}\mathbf{J}$$

for symplectic matrices. It is now straightforward to show [compare Eq. (2.14)]

$$\begin{aligned}\tilde{\mathbf{M}}\mathbf{J}\mathbf{M}&=(-\mathbf{JM}^{-1}\mathbf{J})\mathbf{J}(-\mathbf{J}\tilde{\mathbf{M}}^{-1}\mathbf{J})\\ &=-\mathbf{JM}^{-1}\mathbf{J}\tilde{\mathbf{M}}^{-1}\mathbf{J}=-\mathbf{J}^3=\mathbf{J} ,\end{aligned}$$

where we have used $\mathbf{J}^2=-\mathbf{1}$ and the fact that $\mathbf{M}^{-1}$ is also symplectic if $\mathbf{M}$ is.

Now consider more general functions on the phase space. In particular, using the above notation we see that we can write the symbol $\mathbf{D}(x,k)$ as

$$\mathbf{D}(\mathbf{z})\equiv\int d^8\zeta\, e^{i\zeta\cdot\mathbf{z}}\overline{\mathbf{D}}(\zeta) . \tag{B4}$$

Carrying out the canonical transformation $\mathbf{z}\to\mathbf{z}'$ we find

$$\begin{aligned}\mathbf{D}(z')&\equiv\int d^8\zeta\, e^{i\zeta\cdot\mathbf{z}'}\overline{\mathbf{D}}(\zeta)=\int d^8\zeta\, e^{i\zeta\cdot(\mathbf{Mz})}\overline{\mathbf{D}}(\zeta)\\ &=\int d^8\zeta\, e^{i(\tilde{\mathbf{M}}\zeta)\cdot\mathbf{z}}\overline{\mathbf{D}}(\zeta) .\end{aligned} \tag{B5}$$

Changing the integration variables to $\mathbf{z}'=\mathbf{Mz}$ and using the fact that $\det\mathbf{M}=\pm1$, we get

$$\mathbf{D}(z')=\pm\int d^8\zeta' e^{i\zeta'\cdot\mathbf{z}}\overline{\mathbf{D}}(\mathbf{M}^{-1}\zeta')\equiv\int d^8\zeta' e^{i\zeta'\cdot\mathbf{z}}\overline{\mathbf{D}}'(\zeta') .$$

Now consider the pseudodifferential operator associated with $\mathbf{D}$. Once again, we consider operators linear in z first:

$$f(\hat{z})\equiv\mathbf{a}\cdot\hat{z} .$$

The metaplectic transformation $\hat{\mathcal{M}}(\mathbf{M})$ associated with the linear canonical transformation $\mathbf{M}$ has the following effect upon f:

$$\begin{aligned}\hat{\mathcal{M}}(\mathbf{M})f(\hat{z})\hat{\mathcal{M}}^{\dagger}(\mathbf{M})&\equiv\hat{\mathcal{M}}(\mathbf{M})(\mathbf{a}\cdot\hat{z})\hat{\mathcal{M}}^{\dagger}(\mathbf{M})\\ &=\mathbf{a}\cdot[\hat{\mathcal{M}}(\mathbf{M})\hat{z}\hat{\mathcal{M}}^{\dagger}(\mathbf{M})]=\mathbf{a}\cdot\hat{z}' .\end{aligned}$$

Now using the fact that

$$\hat{z}'=\mathbf{M}\hat{z} ,$$

we see

$$\hat{\mathcal{M}}(\mathbf{M})f(\hat{z})\hat{\mathcal{M}}^{\dagger}(\mathbf{M})=\mathbf{a}\cdot\hat{z}'=\mathbf{a}\cdot(\mathbf{M}\hat{z})=(\tilde{\mathbf{M}}\mathbf{a})\cdot\hat{z} .$$

Thus the transformation properties of the operator $\hat{f}$ are entirely analogous to the linear function f, as desired. Now consider the pseudodifferential operator associated with the symbol (B4). Using the more compact notation this can be written as

$$\mathcal{D}(\hat{z}) \equiv \int d^8\zeta\, \overline{\mathbf{D}}(\zeta) e^{i\zeta\cdot\hat{z}} . \tag{B6}$$

The metaplectic transformation $\hat{\mathcal{M}}(\mathbf{M})$ has the following effect upon $\mathcal{D}(\hat{z})$:

$$\hat{\mathcal{M}}(\mathbf{M})\mathcal{D}(\hat{z})\hat{\mathcal{M}}^\dagger(\mathbf{M}) = \int d^8\zeta\, \overline{\mathbf{D}}(\zeta)\hat{\mathcal{M}}(\mathbf{M}) e^{i\zeta\cdot\hat{z}}\hat{\mathcal{M}}^\dagger(\mathbf{M})$$
$$= \int d^4\zeta\, \overline{\mathbf{D}}(\zeta) e^{i\hat{\mathcal{M}}(\mathbf{M})\zeta\cdot\hat{z}\hat{\mathcal{M}}^\dagger(\mathbf{M})} ,$$

where we have used the unitarity of $\hat{\mathcal{M}}$ to move the operators into the exponent. This can now be written as

$$\hat{\mathcal{M}}(\mathbf{M})\mathcal{D}(\hat{z})\hat{\mathcal{M}}^\dagger(\mathbf{M}) = \int d^8\zeta\, \overline{\mathbf{D}}(\zeta) e^{i\zeta\cdot\hat{z}'}$$
$$= \int d^8\zeta' \overline{\mathbf{D}}'(\zeta') e^{i\zeta'\cdot\hat{z}} ,$$
$$\overline{\mathbf{D}}'(\zeta') = \overline{\mathbf{D}}(\tilde{\mathbf{M}}^{-1}\zeta') ,$$

as with the classical transformation.

APPENDIX C: CALCULATION OF THE p-SPACE WAVE FUNCTIONS IN THE CONVERSION REGION

It is required to evaluate the following integrals:

$$Z_a(p) = \frac{1}{(2\pi)^2} \int d^4q\, e^{-ip\cdot q} Z_a(q) ,$$
$$Z_b(p) = \frac{1}{(2\pi)^2} \int d^4q\, e^{-ip\cdot q} Z_b(q) . \tag{C1}$$

Consider $Z_a(p)$ first:

$$Z_a(p) = \frac{1}{(2\pi)^2} \left[\int_{-\infty}^{0} dq_1 e^{-ip_1q_1} |q_1|^{-i|\eta|^2} \int d^3\mathbf{q}\, e^{-i\mathbf{p}\cdot\mathbf{q}} \alpha_-(\mathbf{q}) \right]$$
$$+ \frac{1}{(2\pi)^2} \left[+ \int_0^\infty dq_1 e^{-ip_1q_1} q_1^{-i|\eta|^2} \int d^3\mathbf{q}\, e^{-i\mathbf{p}\cdot\mathbf{q}} \alpha_+(\mathbf{q}) \right]$$
$$= \frac{1}{(2\pi)^{1/2}} \left[\alpha_-(\mathbf{p}) \int_{-\infty}^{0} dq_1 e^{-ip_1q_1} |q_1|^{-i|\eta|^2} + \alpha_+(\mathbf{p}) \int_0^\infty dq_1 e^{-ip_1q_1} q_1^{-i|\eta|^2} \right] .$$

Here $\alpha_\pm(\mathbf{p})$ are the Fourier transforms of $\alpha_\pm(\mathbf{q})$. Changing variables in the first integral from q_1 to $-q_1$ this becomes

$$Z_a(p) = \frac{1}{(2\pi)^{1/2}} \left[\alpha_-(\mathbf{p}) \int_0^\infty dq_1 e^{ip_1q_1} q_1^{-i|\eta|^2} + \alpha_+(\mathbf{p}) \int_0^\infty dq_1 e^{-ip_1q_1} q_1^{-i|\eta|^2} \right] . \tag{C2}$$

Consider the following integrals:

$$I^\pm(p_1) \equiv \frac{1}{(2\pi)^{1/2}} \int_0^\infty dq_1 e^{\pm ip_1q_1} q_1^{-i|\eta|^2} .$$

We split the real p_1 axis into positive and negative halves:

$$I^\pm(p_1) \equiv \begin{cases} \dfrac{1}{(2\pi)^{1/2}} \displaystyle\int_0^\infty dq_1 e^{\pm ip_1q_1} q_1^{-i|\eta|^2}, & p_1 > 0 \\ \dfrac{1}{(2\pi)^{1/2}} \displaystyle\int_0^\infty dq_1 e^{\pm i(-|p_1|q_1)} q_1^{-i|\eta|^2} , & p_1 < 0 . \end{cases}$$

This shows that $I^+(-p_1) = I^-(p_1)$. Consider $I^\pm(p_1)$ for $p_1 > 0$. Changing variables to $\lambda = p_1 q_1$ we have

$$I^\pm(p_1) = (p_1)^{i|\eta|^2-1} \int_0^\infty d\lambda\, e^{\pm i\lambda} \lambda^{-i|\eta|^2}, \quad p_1 > 0 .$$

Now making a further change of variables in I^+ (I^-) to $\lambda = e^{i\pi/2}t$ ($\lambda = e^{-i\pi/2}t$) leads to

$$I^+(p_1) = p_1^{i|\eta|^2-1} (e^{i(\pi/2)})^{1-i|\eta|^2} \int_0^\infty dt\, e^{-t} t^{-i|\eta|^2}$$
$$= ie^{\pi|\eta|^2/2} p_1^{i|\eta|^2-1} \Gamma(-i|\eta|^2+1) ,$$
$$I^-(p_1) = p_1^{i|\eta|^2-1} (e^{-i(\pi/2)})^{1-i|\eta|^2} \int_0^\infty dt\, e^{-t} t^{-i|\eta|^2}$$
$$= -ie^{-\pi|\eta|^2/2} p_1^{i|\eta|^2-1} \Gamma(-i|\eta|^2+1) ,$$
$$p_1 > 0 .$$

Here Γ is the complex gamma function [15]. Using this result, and $I^+(-p_1) = I^-(p_1)$, in (C2) gives

$$Z_a(p) = \begin{cases} \dfrac{p_1^{i|\eta|^2-1} \Gamma(-i|\eta|^2+1)}{(2\pi)^{1/2}} \left[-i\alpha_-(\mathbf{p}) e^{\pi|\eta|^2/2} + i\alpha_+(\mathbf{p}) e^{-\pi|\eta|^2/2} \right], & p_1 > 0 \\ \dfrac{|p_1|^{i|\eta|^2-1} \Gamma(-i|\eta|^2+1)}{(2\pi)^{1/2}} \left[i\alpha_-(\mathbf{p}) e^{-\pi|\eta|^2/2} - i\alpha_+(\mathbf{p}) e^{\pi|\eta|^2/2} \right], & p_1 < 0 . \end{cases}$$

Using the same analysis to evaluate $Z_b(p)$ leads to

$$Z_b(p)=\begin{cases}\dfrac{\eta^* p_1^{i|\eta|^2}\Gamma(-i|\eta|^2)}{(2\pi)^{1/2}}[\alpha_-(\mathbf{p})e^{\pi|\eta|^2/2}-\alpha_+(\mathbf{p})e^{-\pi|\eta|^2/2}], & p_1>0\\ \dfrac{\eta^* |p_1|^{i|\eta|^2}\Gamma(-i|\eta|^2)}{(2\pi)^{1/2}}[\alpha_-(\mathbf{p})e^{-\pi|\eta|^2/2}-\alpha_+(\mathbf{p})e^{\pi|\eta|^2/2}], & p_1<0\ ,\end{cases}$$

as stated in the text.

APPENDIX D: INVARIANCE OF THE WIGNER TENSOR UNDER CANONICAL TRANSFORMATIONS

The Wigner tensor is invariant under canonical transformations. This is not obvious from Eq. (2.4), but it helps if we use the fact that the Wigner function of a wave function $|\psi\rangle$ can be written as [2,16]

$$W(x,k)=\frac{1}{(2\pi)^N}\mathrm{Tr}\left[|\psi\rangle\langle\psi|\int d^4\sigma\, d^4\tau \exp[i(\hat{x}-x)\cdot\sigma+i(\hat{k}-k)\cdot\tau]\right] .$$

(We wish to thank Jim Morehead for pointing out this argument and the reference to Berry's work.) This object is obviously invariant under unitary transformations of the Hilbert space because it is the trace of an operator. We can also show it is invariant under classical linear canonical transformations using arguments discussed in Appendix B. What remains is to show that this is in fact the Wigner function of ψ. Using properties of the trace we can rewrite the above definition as

$$\begin{aligned}W(x,k)&=\frac{1}{(2\pi)^N}\langle\psi|\int d^4\sigma\, d^4\tau \exp[i(\hat{x}-x)\cdot\sigma+i(\hat{k}-k)\cdot\tau]|\psi\rangle .\\ &=\frac{1}{(2\pi)^N}\int d^4\sigma\, d^4\tau\, e^{-i(x\cdot\sigma+k\cdot\tau)}\langle\psi|\exp(i\hat{x}\cdot\sigma+i\hat{k}\cdot\tau)|\psi\rangle .\end{aligned}$$

We now make use of Glauber's theorem [17] to simplify the matrix element: if two operators A and B commute with their commutator $[A,[A,B]]=[B,[A,B]]=0$, then

$$e^{A+B}=e^{A}e^{B}e^{-[A,B]/2} .$$

Therefore

$$W(x,k)=\frac{1}{(2\pi)^N}\int d^4\sigma\, d^4\tau\, e^{-i[x\cdot\sigma+k\cdot\tau-(1/2)\sigma\cdot\tau]}\langle\psi|\exp(i\hat{x}\cdot\sigma)\exp(i\hat{k}\cdot\tau)|\psi\rangle .$$

Inserting a complete set of states $|x'\rangle$ between the operator product allows one to do the σ integral. This gives

$$W(x,k)=\int d^4\tau\, d^4x'\, e^{-k\cdot\tau}\delta\left[x-x'-\frac{\tau}{2}\right]\psi^*(x')\exp(\tau\cdot\vec{\partial}_x)\psi(x') ,$$

which, after performing the x' integral and using $\exp(\tau\cdot\partial)\psi(x')=\psi(x'+\tau)$, leads to the standard expression

$$W(x,k)=\int d^4\tau\, e^{-ik\cdot\tau}\psi^*\left[x-\frac{\tau}{2}\right]\psi\left[x+\frac{\tau}{2}\right] .$$

The generalization from the scalar wave case to the Wigner tensor is obvious.

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